# Contribution of Mahaviracharya in the development of theory of Series

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In the present paper, an attempt has been made to summarize some of the salient features of the work of the great ancient Indian Mathematician Mahaviracharya (850 A.D.) on the development of theory of series as evinced from his renowned mathematical text Ganita Sarasangraha. No doubt his predecessors Aryabhata I (476 A.D.) and Brahmagupta (599 A.D.) had their contributions to the subject, yet Mahaviracharya can be named as the first amongst them who put the subject elaborately using lucid methods and charming language.

The text GSS consists of nine chapters but it is only chapters II, III and VI which contain the sutras regarding series. In chapters II the A.P. and G P. are given in detail. For example, the following sutra gives the sum of the A.P. whose first term, common difference and number of terms are known.[1]

रूपेणोनो गच्छो दलीकृत: प्रचयताडितो मिश्र: ।
प्रभवेण पदाभ्यस्त: सङ्कलितं भवति सर्वेषाम् ॥

Algebraically if a=first term, d=common difference and n=number of terms and s=the sum of the series then

$$s = \frac{n}{2}\left[(n-1)\, d+2a\right]$$

The above formula has been given in three ways[2,3,4]. In the following sutra the method is given to find out the number of terms of the series if the first term, common difference and the sum of the series be known.[5]

अष्टोत्तरगुणराशेर्द्विगुणाद्युत्तरविशेषकृतिसहितात् ।
मूलं चययुतमर्धितमाद्यूनं चयहृतं गच्छ: ॥

Symbolically, if a=first term, d=common difference, S=sum of the series and n=number of terms then

$$n= \frac{\sqrt{(2a-d)^2+8d^2s}-2a+d}{2d^2}$$

Note :—For references : See Ganita Sarasangraha by Sh. L. C. Jain

| | | Ch. | p. | Sloka |
|---|---|---|---|---|
| 1. | GSS | 2 | 20 | 61 |
| 2. | GSS | 2 | 20 | 62 |
| 3. | GSS | 2 | 21 | 63 |
| 4. | GSS | 2 | 21 | 64 |
| 5. | GSS | 2 | 22 | 69 |

Apart from the above formula, methods are given [1, 2, 3, 4] to find the common difference and the first term if the remaining term are known. Quite a good number of examples [5, 6] are also given whose solution by the above formula can easily be done. Three rules giving stanzas for splitting up (into the component elements) such as sum of the series (in A. P.) as is combined with the first term (आदि मिश्रधन or with the common difference (उत्तर मिश्रधन) or with the number of terms (गच्छमिश्रधन) or with all these (सर्वमिश्रधन) are given below :-

उत्तरधनेन रहितं गच्छेनैकेन संयुतेन हृतम् ।
मिश्रधनं प्रभवःस्यादिति गणकशिरोमणे विद्धि ॥

"O crest jewel of calculators, understand that misradhana diminished by the Uttardhana and (then) divided by the number of terms increased by one, gives rise to the first term."[7]

Symbolically, if S=sum, a=first term, d=C. D. and n=number of terms then

$$a=\frac{S'-\frac{n\,(n-1)}{2}d}{n+1} \qquad \text{where } S'=S+a.$$

Now in the second stanza

आदिधनोनं मिश्रं रूपोनपदार्धगुणितगच्छेन ।
सैकेन हृतं प्रचयो गच्छविधानात्पदं मुखे सैके ॥

"The misradhana diminished by the ādidhana, and then divided by the (quantity obtained by the) addition of *one* to the (product of the) number of terms multiplied by the half of the number of terms lessened by *one* (gives rise to the common difference). (In splitting of the number of terms from the misradhana) the (required) number of terms (is obtained) in accordance with the rule for obtaining the number of terms, provided that the first term is taken to be increased by *one* (so as to cause a corresponding increase in all the terms)".[8]

Algebraically if S″=S+d=Uttardhana and na=adidhana then

$$d=\frac{S''-na}{\frac{n(n-1)}{2}+1}$$

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|---|---|---|---|---|
| 1. | GSS | 2 | 23 | 73 |
| 2. | GSS | 2 | 23 | 74 |
| 3. | GSS | 2 | 23 | 75 |
| 4. | GSS | 2 | 24 | 76 |
| 5. | GSS | 2 | 23 | 71 |
| 6. | GSS | 2 | 24 | 77 |
| 7. | GSS | 2 | 24 | 80 |
| 8. | GSS | 2 | 25 | 81 |

And in the third stanza

मिश्रादपनीतेष्टौ मुखगच्छौ प्रचयमिश्रविधिलब्धः ।
यो राशिः स चयः स्यात्करणमिदं सर्वसंयोगे ॥

"The misradhan is diminished by the first term and the number of terms, both (of these) being optionally chosen ; (then) that quantity, which is obtained (from this difference) by applying the rule for (splitting up) the Uttarmisradhana happens to be the common difference (required here). This is the method of work in (splitting up) the all combined (misradhana)".[1]

Symbolically, if $\overline{S}=S+n$

$$S=a+(a+d)+(a+2d)+\ldots\ldots\text{to n terms.}$$

then $$\overline{S}=(a+1)\,\overline{+}\,(a+1+d)\,\overline{+}\,(a+1+2d)+\ldots\ldots\text{to n terms}$$

$$=\frac{n}{2}\Big[2(a+1)+(n-1)\ d\Big]$$

which is a quadratic equation and hence n can be found. Now according to the above rule, a and n can be chosen in any way. This method is the same as the previous one.

**Example[2] :**

द्वित्रिकपंचदशाग्रा चत्वारिंशन्मुखादिमिश्रधनम् ।
तत्र प्रभवं प्रचयं गच्छं सर्वं च मे ब्रूहि ॥

Forty, exceeded by 2, 3, 5 and 10, represents (in order) the adimisradhana and the other (misradhanas). Tell me what (respectively) in these cases happens to be the first term, the common difference, the number of terms and all (these three)."

This means

(i) find a when $S'=42$, $d=3$, $n=5$.

(ii) find d when $S''=43$, $a=2$, $n=5$.

(iii) find n when $\overline{S}=45$, $a=2$, $d=3$, and

(iv) find a, d, n when $S=50$.

From the formulae given above the results can be obtained easily. In the following sutra the rule is given for finding, in relation to two (series), the number of terms wherein are optionally chosen their mutually interchanged first terms and common difference as also their sums which may be equal or (one of which may be) twice, thrice, half or one-third or any such (multiple or fraction of the other) :[3]

व्येकात्महतो गच्छः स्वेष्टघ्नो द्विगुणितान्यपदहीनः ।
मुखमात्मोनान्यकृतिद्विकेष्टपदघातवर्जिता प्रचयः ॥

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1. GSS 2 25 82
2. GSS 2 25 83
3. GSS 2 26 86

"The number of terms (in one series), multiplied by itself as lessened by *one* and then multiplied by the chosen ratio between the sums of the two series, and then diminished by twice the number of terms in the other series gives (rise to the interchangeable) first term of one (of the series). The square of the (number of terms in the) other (series) diminished (again) by the product of two (times the) chosen (ratio) and the number of terms (in the first series) gives (rise to the interchangeable common difference (of that series)."

Symbolically if $S$, $S_1$ be the sums, $a$, $a_1$ the first terms and $d$, $d_1$ the common differences of the two given series then $a_1=\frac{S_1}{S}a$ and $d_1=\frac{S_1}{S}d$. Now if $\frac{S_1}{S}=r$ and $n$, $n_1$ be the respective number of terms in the two series, then according to the above formula

$$a=n\,(n-1)\times r-2n_1$$

and

$$d=(n_1)^2-n_1-2rn$$

**Example**[1]:

पंचाष्टगच्छपुंसोर्व्यस्तप्रभवोत्तरे समानधनम् ।
द्वित्रिगुणादिधनं वा ब्रूहि त्वं गणक विगणय्य ।।

"In relation to two men (whose wealth is measured) respectively by the sums of two series in A.P. having 5 and 8 for the number of terms, the first term and the common difference and both these series be interchangeable (in relation to each other), the sums (of the series) being equal or the sum (of one of them) being twice, thrice, or any such (multiple of that of the other), 'O arithmetician, give out the (value of these) sums and the interchangeable first term and common difference after calculating (them all) well."

**Solution :** If $S=S_1$ then $r=1$

so in the above case where $n=5$ and $n_1=8$ we have

$$a=n\,(n-1)\times 1-2n_1$$
$$=5\,(5-1)\times 1-2\times 8=20-16=4$$

and

$$d=(n_1)^2-n_1-2\,rn$$
$$=(8)^2-8-2\times 1\times 5=64-8-10=46$$

Then

$$S=\frac{5}{2}\Big(2\times 4+(5-1)\times 46\Big)=5\,(4+92)=480$$

and

$$S_1=\frac{8}{2}\Big(2\times 46+(8-1)\times 4\Big)=4\,(9\;+28)=480$$

which proves that

$$r=\frac{S_1}{S}=1.$$

1. GSS 2 27 87

*Partial Sums*

The sum of any part of a series is known as the partial sum of the series. In the following verse, the method is given for finding the partial sum of a given series :[1]

सपदेष्टं स्वेष्टमपि व्येकं दलितं चयाहतं समुखम् ।
शेषेष्टगच्छगुणितं व्युत्कलितं स्वेष्टवित्तं च ॥

"(Take) the chosen off number of terms as combined with the total number of terms (in the series) and (take) also your own chosen off number of terms (simply) diminish (each of) these (resulting products) and these (resulting quantities) when multiplied by the remaining number of terms (respectively), give rise to the sum of the remainder series and to the sum of the chosen off part of the series (in order)."

Symbolically, Vyutkalita$=S_v$

$$=\left[\frac{n+p-1}{2}d+a\right](n-p)$$

and the sum of the chosen part$=S_p$

$$=\left[\frac{p-1}{2}d+a\right]p.$$

where $p$ is the number of terms of the chosen part of the series. Another form of the same formula is given in a different verse.[2]

In the following sutra is given the rule for finding the sum of a series in arithmetic progression in which the common difference is either positive or negative :[3]

व्येकार्धपदोनाधिकचयघातोनान्वितः पुनः प्रभवः ।
गच्छाभ्यस्तो हीनाधिकचयसमुदायसंकलितम् ॥

"The first term is either decreased or increased by the product of the negative or the positive common difference and the quantity obtained by halving the number of terms in the series as diminished by one. (Then), this is (further) multiplied by the number of terms of the series and (thus), the sum of series of terms in arithmetical progression with positive or negative common difference is obtained."

Symbolically, $S=\left(\pm\frac{n-1}{2}d+a\right)n$

where a, d, n and S have their usual meanings.

**Example[4] :**

चतुरुत्तरदश चादिर्हीनचयस्त्रीणि पञ्च गच्छः किम् ।
द्वावादिर्वृद्धिचयः षट् पदमष्टौ धनं भवेदत्र ॥

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|---|---|---|---|---|
| 1. | GSS | 2 | 32 | 106 |
| 2. | GSS | 2 | 33 | 107 |
| 3. | GSS | 6 | 165 | 290 |
| 4. | GSS | 6 | 165 | 291 |

The first term is 14, the negative common difference is 3 and the number of terms is 5 ; the first term is 2 ; the positive common difference is 6 and the number of terms is 8. What is the sum of the series in (each of) these cases ?

**Solution :** (i) $a=14, d=-3, n=5.$

$$\therefore S_1=5\left[\frac{5-1}{2}\times(-3)+14\right]=5(-6+14)=5\times 8=40$$

(ii) $a=2, d=6, n=8$

$$\therefore S_2=8\left[\frac{8-1}{2}\times(6)+2\right]=8(21+2)=8\times 23=184$$

In the following sutra the rule is given for finding the time of *arrival* of two persons at a common terminus when one, who is moving (with successive velocities representable) in arithmetical progression and another moving with steady unchanging velocity, may meet together again (after starting at the same instant of time) :[1]

ध्रुवगतिरादिविहीनश्चयदलभक्तः सरूपकः कालः ।
द्विगुणो मार्गस्तद्गतियोगहृतो योगकालः स्यात् ॥

"The unchanging velocity is diminished by the first term (of the velocities in series in A. P.) and is (then) divided by the half of the common difference. On adding *one* (to the resulting quantity), the required time (of meeting) is arrived at. (Where two persons travel in opposite directions, each with a definite velocity) twice (the average distance to be covered by either of them) is the (whole) way (to be travelled).

This when divided by the sum of their velocities gives rise to the time of (their) meeting."

Symbolically, if $V$ = the unchanging velocity

$a$ = first term of changing vel.

$d$ = common difference

$t$ = time taken.

then $t = (V-a) \div \frac{d}{2} + 1.$

**Example**[2] **:**

कश्चिन्नरः प्रयाति त्रिभिरादा उत्तरैस्तथाष्टाभिः ।
नियतगतिरेकविंशतिरनयोः कः प्राप्तकालः स्यात् ॥

"A certain person goes with velocity 3 in the beginning increased (regularly) by 8 as the (successive) C. D. The steady unchanging velocity (of another person) is 21. What may be the time of their meeting (again if they start from the same place, at the same time, and move in the same direction)?"

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|---|---|---|---|---|
| 1. | GSS | 6 | 173 | 319 |
| 2. | GSS | 6 | 174 | 320 |

**Solution** : $V = 21, \quad a = 3, \quad d = 8$

then $\quad t = (V-a) \div \frac{d}{2} + 1$

$$= (21-3) \div \frac{8}{2} + 1$$

$$= 18 \div 4 + 1 = \frac{11}{2}.$$

In the following stanza the rule is given for arriving at the time and distance of meeting together (when two persons start from the same place, at the same time and travel) with (varying) velocities in A. P.[1]

उभयोराद्यो: शेषश्चयशेषहृतोद्विसङ्गुण: सैक: ।
युगपत्प्रयाणयो: स्यान्मार्गे तु समागम: काल: ॥

"The difference between the first two terms divided by the difference between the two common differences when multiplied by *two* and increased by *one*, gives rise to the time of coming together on the way by the two persons travelling, simultaneously (with two series of velocities varying in A. P.)"

Symbolically, if $a$, $a_1$ be the velocities in beginning and $d$, $d_1$ be their respective common differences then the time of meeting is given by

$$t = \frac{a_1 \sim a_2}{d_1 - d_2} \times 2 + 1$$

The same formula has been given in another stanza[2] too.

**Example**[3] **:**

पंचाद्यष्टोत्तरत: प्रथमो नाथ द्वितीयनर: ।
आदि: पञ्चघ्ननव प्रचयो हीनोऽष्ट योगकाल: क: ॥

"The first man travels with velocity beginning with 5, and increased (successively) by 8 as the common difference. In the case of the second person, the starting velocity is 45, and the common difference is *minus* 8. What is the time of meeting ?"

**Solution :** $t = \frac{5 \sim 45}{8-(-8)} \times 2 + 1 = 5 + 1 = 6.$

In the following sutra the rule is given for arriving at the number of bricks to be found in structures made up of layers (of bricks one over the other).[4]

तरवर्गो रूपोनस्त्रिभिर्विभक्तस्तरेण संगुणित: ।
तरसंकलिते स्वेष्टप्रताडिते मिश्रत: सारम् ॥

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|---|---|---|---|---|
| 1. | GSS | 6 | 174 | $322\frac{1}{2}$ |
| 2. | GSS | 6 | 175 | $324\frac{1}{2}$ |
| 3. | GSS | 6 | 175 | $325\frac{1}{2}$ |
| 4. | GSS | 6 | 176 | $330\frac{1}{2}$ |

"The square of the number of layers is diminished by *one*, divided by *three*, and (then) multiplied by the number of layers. On adding (to quantity so obtained) the product, obtained by multiplying the arbitrarily chosen number (representing) the bricks in (the topmost layer) by the sum of the (natural numbers beginning with one and going upto the given) number of layers, the required answer is obtained".

Symbolically, if $n$ be the number of layers, and $a$ number arbitrarily chosen representing the bricks in the topmost layer, then

$$\text{Total number of bricks} = \frac{n^2-1}{3} \times n + a \times \frac{n\,(n+1)}{2}$$

**Example**[1] **:**

पञ्चतरंकेनाग्रं व्यवघटिता गणितविन्मिश्रे ।
समचतुरश्रश्रेढी कतीष्टकाः स्युर्ममाचक्ष्व ॥

"There is constructed an equilateral quadrilateral structure consisting of 5 layers. The topmost layer is made up of one brick. O' you, who know the calculation tell me how many bricks there are (in all)".

**Solution :** $n = 5, \quad a = 1.$

$$\text{So total number of bricks} = \frac{5^2-1}{3} \times 5 + 1 \times \frac{5\,(5+1)}{2}$$

$$= \frac{25-1}{3} \times 5 + \frac{5 \times 6}{2}$$

$$= 40 + 15$$

$$= 55 \text{ bricks.}$$

Now we shall consider the work of Mahavira on Geometrical progressions. In the following sutra is given the rule for finding gunadhana (गुणधन) and the sum of a G. P. if the first term, common ratio and the number of terms of the series are known :[2]

पदमितगुणहतिगुणितप्रभवः स्याद्गुणधनं तदाद्यूनम् ।
एकोनगुणविभक्तं गुणसङ्कलितं विजानीयात् ॥

The product of the first term with the common ratio multiplied to itself as many times as the number of terms gives the gunadhana. It be known that the gunadhana lessened by the first term and divided by one less than the number of terms gives the gunasankalita.

Symbolically, if $n$ = the number of terms

$a$ = first term

and $r$ = common ratio then

gunadhana $= ar^n = (n+1)^{th}$ term.

and gunasankalita (sum of the series) $= S = \dfrac{ar^n - a}{r-1}$

1. GSS 6 177 331½
2. GSS 2 28 93

In the following sutra another rule is given to find out the sum of a series in G. P.[1]

समदलविषमखरूपो गुणगुणितो वर्गताडितो गच्छ: ।
रूपोन: प्रभवघ्नो व्येकोत्तरभाजित: सारम् ।।

"The number of terms in the series is caused to be marked (in a separate column) by *zero* and by *one* (respectively) corresponding to the even (value) which is halved and to the uneven (value from which *one* is substracted till by continuing these processes *zero* is ultimately reached), then this (representative series made up of zero and one is used in order from the last *one* there in, so that this *one* multiplied by the common ratio is again) multiplied by the common ratio (wherever *one* happens to be the denoting item) and multiplied so as to obtain the square (wherever *zero* happens to be the denoting item). When (the result of) this (operation) is diminished by *one* and (is then) multiplied by the first term, and (is then) divided by the common ratio lessened by *one* it becomes the sum (of the series)."

**Example**[2] :

स्वर्णद्वयं गृहीत्वा त्रिगुणधनं प्रतिपुरं समार्जयति ।
य: पुरुषोऽष्टनगर्यां तस्य कियद्वित्तमाचक्ष्व ।।

"Having obtained 2 gold coins (in some city), a man goes on from city to city, earning (everywhere) three times (of what he earned immediately before). Say how much he will make in the eighth city."

**Solution** : Here $n = 7$, $r = 3$, $a = 2$

$7 =$ an odd number, hence *one* is subtracted from it and also it is denoted by *one*.

$7 - 1 = 6 =$ an even number, hence it is divided by 2 and 0 denotes it

$\frac{6}{2} = 3 =$ an odd number, it is diminished by *one* and 1 denotes it

$3 - 1 = 2 =$ an even number, it is divided by 2 and 0 denotes it

$\frac{2}{2} = 1 =$ an odd number, it is diminished by *one* and 1 denotes it

$1 - 1 = 0 =$, where the operation ends.

Now the whole is put in the side column. Since in the column, 1 is in the last hence it is multiplied by the common ratio 3, then comes zero so 3 is squared and we get $3^2$, then comes 1 above it so it is multiplied by 3 *i.e.* we get $3^3$, then comes zero above it so it is squared and we get $3^6$, then in the end there is *one* above it so it is multiplied by 3 and get $3^7$. So the guradhana $= ar^n = 2 \times 3^7 = 2 \times 2187 = 4374$ coins, will be the amount obtained by the man in eighth city.

| |
|---|
| 1 |
| 0 |
| 1 |
| 0 |
| 1 |

The rules for finding out the last term and the sum of series in G. P. have also been given in stanza[3]. There are other sutras in which rules have been given to find out the first term, common ratio and the number of terms of the series in G. P. [4], [5], [6], [7].

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|---|---|---|---|---|
| 1. | GSS | 2 | 29 | 94 |
| 2. | GSS | 2 | 30 | 96 |
| 3. | GSS | 2 | 30 | 95 |
| 4. | GSS | 2 | 30 | 97 |
| 5. | GSS | 2 | 30 | 98 |
| 6. | GSS | 2 | 30 | 101 |
| 7. | GSS | 2 | 32 | 103 |

In the following sutras the rules have been given to find out the sum of a series in geometrical progression, wherein the terms are either increased or decreased (in a specified manner by a given known quantity).[1]

गणचितिरन्यादिहृता विपदाधिकहीनसंगुणा भक्ता ।
व्येकगुणेनान्या फलरहिता हीनेऽधिके तु फलयुक्ता ॥

Algebraically, if $S$ = sum of the series, $a$ = first term

$n$ = number of terms, $r$ = common ratio and

$m$ = the quantity to be added or subtracted from each term of the series in G. P., and

$S'$ = the sum of the series in G. P., then

$S$ = sum of the resulting series

$$= \pm \frac{\left(\frac{S'}{a} - n\right) m}{r-1} + S'$$

Proof : *Theorem* : Let

$S = a + (ar \pm m) + [(ar \pm m) r \pm m] + \dots$ to $n$ terms

and $S' = a + ar + ar^2 + ar^3 + \dots$ to $n$ terms

Now

$$S = [a + ar + ar^2 + ar^3 + \dots \text{ to } n \text{ terms}]$$
$$+ m[(r + r^2 + \dots \text{ to } \overline{n-1} \text{ terms})]$$
$$+ m[(r + r^2 + r^3 + \dots \text{ to } \overline{n-2} \text{ terms})] + \dots + m$$

$$= a\frac{r^n-1}{r-1} + m\frac{r^{n-1}-1}{r-1} + m\frac{r^{n-2}-1}{r-1} + \dots + m\frac{r-1}{r-1}$$

$$= S' + \frac{m}{r-1}[(r^{n-1}-1) + (r^{n-2}-1) + (r^{n-3}-1) + \dots + (r-1)]$$

$$= S' + \frac{m}{r-1}[(r+r^2+r^3+\dots+r^{n-1}) - 1 \times (n-1)]$$

$$= S' + \frac{m}{r-1}\left[r.\frac{r^{n-1}-1}{r-1} - (n-1)\right]$$

$$= S' + \frac{m}{r-1}\left[\frac{r^n-1}{r-1} - 1 - n + 1\right]$$

$$= S' + \frac{m}{r-1}\left[\frac{S'}{a} - n\right]$$

1. GSS 6 172 314

which can be generalised as $S = \pm \dfrac{\left(\dfrac{S'}{a} - n\right) m}{r-1} + S'$

Now we shall discuss the contribution of Mahavira in the development of the series which can be put in another category called miscellaneous series. This work no doubt, is quite voluminous and it can be said without any hesitation that no other Hindu mathematician contributed so much.

In the following stanza a rule is given for finding the sum of the squares of natural numbers.[1] He has not given any formula for the sum of natural numbers like others.

सैकेष्टकृतिर्द्विघ्ना सैकेष्टोनेष्टदलगुणिता ।
कृतिघनचितिसंघातस्त्रिकभक्तो वर्गसंकलितम् ॥

Algebraically, if $n$ = number of terms and

$$\sum n = \frac{n\,(n+1)}{2} = \text{sum of first } n \text{ natural nos.}$$

$\Sigma\, n^2$ = sum of the squares of $n$ natural nos.

then

$$\frac{1}{3}\Big[\ 2\,(n+1)^2 - (n+1)\ \Big]\,\frac{n}{2} = \Big[\ n^2 + n^3 + \frac{n\,(n+1)}{2}\ \Big]\frac{1}{3} = \sum n^2$$

$$= \frac{n \times (n+1) \times (2n+1)}{6}\,.$$

In the following sutra a rule is given for finding the sum of the squares of numbers which are in A. P. This is most general form of the rule which can be applied broadly.[2]

द्विगुणैकोनपदोत्तरकृतिहतिषष्ठांशमुखचयहतयुतिः ।
व्येकपदघ्ना मुखकृतिसहिता पदताडितेष्टकृतिचितिका ॥

Algebraically, if $a$ = first term, $d$ = common diff.

$n$ = number of terms

and $s$ = sum of the squares of the terms which are in A. P. then

$$S = \sum\Big[a + (n-1)\,d\Big]^2 = n\left[\left\{\frac{(2n-1)\,d^2}{6} + ad\right\}(n-1 + a^2\ \right]$$

which can easily be substantiated by taking LHS.

*i.e.* $\Sigma[a + (n-1)\,d]^2 = \Sigma[(a-d)^2 + 2\,nd\,(a-d) + n^2\,d^2]$

$= n\,(a-d)^2 + 2\,d\,(a-d)\,\Sigma n + d^2\,\Sigma n^2$

we know that $\sum n = \dfrac{n\,(n+1)}{2}$ and $\sum n^2 = \dfrac{n\,(n+1)\,(2n+1)}{6}$

Hence by substituting these values we get the result

1. GSS 6 167 296
2. GSS 6 167 2°8

$$S = n\left[\left\{\frac{(2n-1)\,d^2}{6} + ad\right\}(n-1) + a^2\right]$$

which is in its most general form. Another method of the same formula is given in[1] of the text.

Now comes the rule for finding the sum of the cubes of first $n$ natural numbers which has been given to be equal to square of the sum of first $n$ natural numbers.[2]

गच्छार्धवर्गराशीरूपाधिकगच्छवर्गसङ्गुणितः ।
धनसङ्कलितं प्रोक्तं गणितेऽस्मिन् गणिततत्त्वज्ञैः ॥

"The square of half of the number of terms is multiplied by the square of (the number of term increased by one) which gives rise to the sum of cubes of first $n$ natural numbers as stated by mathematicians."

Algebraically, $\sum n^3 = \left\{\frac{n\,(n+1)}{2}\right\}^2$

In the following stanza he has given a rule for finding out the sum of the cubes of the terms which are in A. P. This formula is in its most general form.[3]

चित्यादिहृतिर्मुखचयशेषघ्ना प्रचयनिघ्नचितिवर्गे ।
आदौ प्रचयादूने वियुता युक्ताधिके तु धनचितिका ॥

Algebraically, if $S$ = sum of terms in A. P.

$a$ = first term, $d$ = common difference

$n$ = number of terms

$S_n$ = sum of the given series then

$$S_n = \Sigma[a + (n-1)\,d]^3 = S^2 d \pm Sa\,(a-d)$$

where $S = \frac{n}{2}\left[\,2a + (n-1)\,d\,\right]$

or specifically (*i*) when $a > d$, $S_n = + Sa\,(a-d) + S^2 d$

(*ii*) when $a < d$, $S_n = - Sa\,(a-d) + S^2 d$

In the following stanza a rule has been given for finding out the sum of such a series whose each term is the sum of an A. P. of natural nos. having the number of terms equal to the term itself.[4]

द्विगुणैकोनपदोत्तरकृतिहतिरङ्गाहृता चयार्धयुता ।
आदिचयाहतियुक्ता व्येकपदघ्नादिगुणितेन ॥
सैकप्रभवेन युता षट्दलगुणितैव चितिचितिका ॥

| | | | | |
|---|---|---|---|---|
| 1. | GSS | 6 | 168 | 299 |
| 2. | GSS | 6 | 168 | 301 |
| 3. | GS3 | 6 | 169 | 303 |
| 4. | GSS | 6 | 169 | 305-305½ |

Symbolically we can write, if $S_a = \frac{a(a+1)}{2}$, $S_{a+d} = \frac{(a+d)(a+d+1)}{2}$ etc.,

$$S = S_a + S_{a+d} + S_{a+2d} + \ldots + S_{a+(n-1)d}$$

$$= \left[ \left\{ \frac{(2n-1)\ d^2}{6} + \frac{d}{2} + ad \right\} \times (n-1) + a(a+1) \right] \times \frac{n}{2}$$

In the following stanza[1] a rule has been given for finding of the sum of the series which can be written symbolically in the form

$1+(1+2)+(1+2+3)+\ldots+(1+2+3+\ldots+n)$, $n^2$, $n^3$ and $\Sigma n$.

$$i.e.\ S = \Sigma n + \Sigma \frac{n\ (n+1)}{2} + n^2 + n^3 .$$

सैकपदार्धपदाहृतिरश्वैर्निहृता पदोनिता त्र्याप्ता ।
सैकपदघ्ना चितिचितिचितिकृतिघनसंयुतिर्भवति ॥

Algebraically,

$$S = \frac{\frac{n\ (n+1) \times 7}{2} - n}{3} \times (n+1)$$

which can be proved easily by substituting values

$$\Sigma\ n = \frac{n\ (n+1)}{2},\ \frac{1}{2}\ \Sigma(n^2+n) = \frac{1}{2}\ \Sigma n^2 + \frac{1}{2}\ \Sigma n$$

$$= \frac{n\ (n+1)\ (2n+1)}{12} + \frac{n\ (n+1)}{4}.$$

Lastly, in the following stanza a rule has been given for finding out a single formula for the sum of the four above mentioned series.[2]

गच्छस्त्रिरूपसहितो गच्छचतुर्भागताडितः सैकः ।
सपदपदकृतिविनिघ्नो भवति हि संघातसंकलितम् ॥

Symbolically, the above formula takes the form

$$\Sigma n^2 + \Sigma n^3 + \Sigma S_n + \Sigma n = \left[ (n+3) \times \frac{n}{4} + 1 \right] (n^2+n)$$

| | | | | |
|---|---|---|---|---|
| 1. | GSS | 6 | 170 | 307½ |
| 2. | GSS | 6 | 171 | 309½ |

where $S_n = S_1 + S_2 + S_3 + S_4 + \dots + S_n$

and $S_n = \frac{n(n+1)}{2} = \Sigma n$

*i.e.* $\Sigma S_n = \Sigma\Sigma n = \left[\left\{\frac{(2n-1)}{6} + \frac{1}{2} + 1\right\}(n-1) + 1\,(1+1)\right]\frac{n}{2}$

Since $a = d = 1$ in the formula $\frac{n}{2}\left[\left\{\frac{(2n-1)\,d^2}{6} + \frac{d}{2} + ad\right\}(n-1) + a(a+1)\right]$

**Example[1] :**

सप्तकृतेः षट्षष्ट्यास्त्रयोदशानां चतुर्दशानां च ।
पंचाग्रविंशतीनां किं स्यात् संघातसंकलितम् ॥

"What would be the (required) collective sum in relation to the (various) series represented by (each of) 49, 66, 13, 14 and 25 ?"

**Solution :**

The above given values are the number of terms in the five series. Hence for the first series in which $n = 49$.

$$\text{Required sum} = \left(\frac{(n+3)\,n}{4} + 1\right)(n^2 + n)$$

$$= \left(\frac{(49+3)\times 49}{4} + 1\right)(49^2 + 49)$$

$$= \left(\frac{52 \times 49}{4} + 1\right) \times 49\,(49 + 1)$$

$$= [\,13 \times 49 + 1\,] \times 49 \times 50$$

$$= [\,637 + 1\,] \times 2450$$

$$= 638 \times 2450$$

$$= 1563100.$$

1. GSS 6 171 $310\frac{1}{2}$